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Non-perturbative effects in matrix models and D-branes

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Abstract: The large order growth of string perturbation theory in $c \leq 1$ conformal field theory coupled to world sheet gravity implies the presence of $O(e^{-\frac{1}{g_s}})$ non-perturbative effects, whose leading behavior can be calculated in the matrix model approach. Recently it was proposed that the same effects should be reproduced by studying certain localized D-branes in Liouville Field Theory, which were constructed by A. and Al. Zamolodchikov. We discuss this correspondence in a number of different cases: unitary minimal models coupled to Liouville, where we compare the continuum analysis to the matrix model results of Eynard and Zinn-Justin, and compact $c = 1$ CFT coupled to Liouville in the presence of a condensate of winding modes, where we derive the matrix model prediction and compare it to Liouville theory. In both cases we find agreement between the two approaches. The $c = 1$ analysis also leads to predictions about properties of D-branes localized in the vicinity of the tip of the cigar in SL(2)/U(1) CFT with $c = 26$.

Keywords: Bosonic Strings, D-branes, Matrix Models.

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JHEP09(2003)057

Contents

1. Introduction

In the late 1980's and early 1990's it was pointed out that low dimensional models of non-critical string theory, corresponding to $c \leq 1$ conformal field theories coupled to world sheet quantum gravity, provide interesting toy models in which one can hope to study non-perturbative effects in string theory in a controlled setting. In particular, in [1, 2, 3] it was shown that certain large-N matrix models in a double scaling limit allow one to efficiently compute amplitudes in these theories to all orders in string perturbation theory. The general structure of the perturbative amplitudes is

$$\mathcal{A} \simeq \sum_{h=0}^{\infty} g_s^{2h-2} c_h \,, \tag{1.1}$$

where the genus h contribution c_h can be computed from the matrix model.

The question of the non-perturbative completion of these perturbative results was raised already in the original papers on the double scaling limit and was studied further in [4, 5, 6] and other papers. The matrix model results for the coefficients c_h showed that the latter grow with the genus like $c_h \simeq a^h (2h)!$; a was found to be positive for unitary models, so that the amplitudes are not Borel summable. Standard results on asymptotic series suggest that in this situation the leading non-perturbative ambiguities are of order $\mathcal{A}_{\rm np} \sim g_s^{f_A} \exp(-f_D/g_s)$, where f_A and f_D are computable from the large order behavior of the perturbative series.

In another significant development, S. Shenker [7] pointed out that the $(2h)!$ large order behavior exhibited by low-dimensional non-critical strings is in fact expected to be generic in string theory, and thus the $\exp(-1/g_s)$ non-perturbative effects are very general as well. A natural source of such effects was found by J. Polchinski [8], who suggested that when going beyond perturbation theory in closed string theory, one has to include contributions of Riemann surfaces with holes, with boundary conditions corresponding to D-instantons – D-branes that are localized in spacetime and have finite action (or disk partition sum). It was shown in [8] that the combinatorics of summing over configurations containing multiple disconnected disks with these boundary conditions is such that the contribution of a single disk has to be exponentiated, leading to a non-perturbative contribution

$$\mathcal{A}_{\rm D} \sim e^{Z_{\rm disk}} \,. \tag{1.2}$$

Here $Z_{\rm disk}$ is the disk partition sum of the D-instanton; it goes like $Z_{\rm disk} \sim 1/g_s$, as required for Shenker's arguments.

The idea that D-instantons provide an important source of non-perturbative effects in string theory proved to be very fruitful in many contexts, but remarkably the original setting in which these effects were first discussed, $c \leq 1$ string theory, remained mysterious for a long time.[1] The main problem was identifying the localized branes that could lead to such effects. Indeed, if one naively tries to construct a localized D-brane in Liouville theory, since the energy of the brane goes like $1/g_s$, and the string coupling depends on the Liouville field ϕ, $g_s \simeq \exp(Q\phi)$, it seems that the D-brane will feel a force pushing it to the strong coupling region $\phi \to \infty$, where it becomes light, and leads to breakdown of perturbation theory. On general grounds one would expect the Liouville potential, $\mu \exp(2b\phi)$ to regularize the problem, but no explicit construction of localized D-branes with sensible properties was known.

Progress came in a beautiful paper by A. and Al. Zamolodchikov [11], who constructed the necessary D-branes in Liouville field theory, and analyzed some of their properties. They showed that these branes correspond to a boundary state obtained by quantization of a classical solution for which $\phi \to \infty$ on the boundary of the world sheet. Nevertheless, correlation functions of bulk and boundary operators on the disk with these boundary conditions are sensible and well behaved.

The open string spectrum corresponding to these D-branes is particularly simple. For the simplest brane, labeled as the $(1,1)$ brane in [11], one finds in the open string sector only the conformal block of the identity. Thus, these branes are very natural candidates for the role of D-instantons in non-critical string theory. Indeed, it was recently proposed [12, 13] that the branes of [11] are the source of the leading non-perturbative effects observed in the matrix model.

[1] But, see [9, 10] for some discussions.

JHEP09(2003)057

This assertion is one of the consequences of the recent progress on the relation between the $c \leq 1$ matrix model and two dimensional string theory [14, 12, 15], according to which the D-branes of [11] are nothing but the eigenvalues of the matrices which figure in the analysis of [1, 2, 3] and the relation between the matrix model and Liouville theory is an example of an open-closed large N duality. In the matrix model, it is known that the leading non-perturbative effects are associated with eigenvalue tunneling (see e.g. [4, 5]); hence it is natural that in the continuum formalism, these effects should be due to the D-branes of [11].

The main purpose of this paper is to study the leading non-perturbative effects in the matrix model solutions of various $c \leq 1$ models coupled to Liouville and, when possible, to compare them to the effects of D-instantons in the continuum formalism, obtained by using the D-branes of [11].

For $c < 1$, the matrix model analysis was done in [4, 5, 6]. We construct the appropriate D-instantons in the continuum approach, and show that they give the same leading non-perturbative contributions as those found in [6] (for the unitary minimal models). This case was discussed in [13]; our results do not agree with those of [13], since we use a different set of Liouville $\times$ matter branes to construct the D-instantons.

For $c = 1$, we study a compact scalar field $x \sim x + 2\pi R$, with a winding mode perturbation $\delta\mathcal{L} = \lambda \cos R(x_L - x_R)$, coupled to world sheet gravity. We derive the leading non-perturbative effects as a function of λ from the matrix model, and compare a small subset of these results to those obtained by using the branes of [11]. In all cases where a comparison can be made, the two approaches agree. We also use the $c = 1$ results to obtain matrix model predictions for non-perturbative effects in Sine-Liouville and the two dimensional euclidean black hole backgrounds.

The paper is organized as follows. In section 2 we discuss unitary $c < 1$ minimal models coupled to gravity. We briefly review the matrix model analysis of [6], and then discuss the form of the relevant D-instantons in the continuum approach. We point out that the quantity calculated in [6] is very natural from the continuum point of view, compute it and show that the continuum result agrees with the matrix model one.

In section 3 we discuss the (T-dual of the) Sine-Gordon model coupled to gravity. Using the results of [16], we derive a differential equation that governs the non-perturbative effects as a function of the Sine-Gordon coupling λ. We solve this equation, and use the solution to obtain information about the leading non-perturbative effects in Sine-Liouville theory, and for a particular value of $R(= 3/2$ of the self dual radius), for the euclidean two dimensional black hole (or cigar) with $c = 26$. We also perform a few continuum (Liouville) calculations, which reproduce some aspects of the matrix model analysis.

In section 4 we conclude and comment on our results. A few appendices contain useful technical results.

2. Non-perturbative effects in unitary minimal models coupled to gravity

In this section we study the leading non-perturbative effects in minimal models coupled to gravity. We restrict the discussion to the case of unitary minimal models, since the matrix model analysis for this case was already done in [6]. It should be straightforward to generalize our results to the non-unitary minimal model case.

JHEP09(2003)057

Minimal models (for a review see e.g. [17]) are labeled by two integers (p,p'). The unitary case corresponds to $p' = p+1$. The central charge of the $(p,p+1)$ model is given by

$$c = 1 - \frac{6}{p(p+1)}\,. \tag{2.1}$$

The minimal model contains a finite number of primaries of the Virasoro algebra labeled by two integers, (m,n), whose dimensions are

$$h_{m,n} = \frac{[(p+1)m - pn]^2 - 1}{4p(p+1)}\,, \tag{2.2}$$

where the labels (m,n) run over the range $m = 1,2,\ldots,p-1$, $n = 1,2,\ldots,p$, and one identifies the pairs (m,n) and $(p-m,p+1-n)$.

The coupling to gravity leads in conformal gauge to the appearance of the Liouville field, ϕ, which is governed by the action

$$S_L = \int \frac{d^2\sigma}{4\pi}\left((\partial\phi)^2 + Q\hat{R}\phi + \mu_{_L} e^{2b\phi}\right). \tag{2.3}$$

The central charge of the Liouville model is given by

$$c = 1 + 6Q^2 \tag{2.4}$$

and the parameter b is related to Q via the relation

$$Q = b + \frac{1}{b}\,. \tag{2.5}$$

In general, b and Q are determined by the requirement that the total central charge of matter and Liouville is equal to 26. In our case, (2.1) and (2.4) imply that

$$b = \sqrt{\frac{p}{p+1}}\,. \tag{2.6}$$

An important class of conformal primaries in Liouville theory corresponds to the operators

$$V_\alpha(\phi) = e^{2\alpha\phi} \tag{2.7}$$

whose scaling dimension is given by $\Delta_\alpha = \bar{\Delta}_\alpha = \alpha(Q-\alpha)$. The Liouville interaction in (2.3) is $\delta\mathcal{L} = \mu_{_L} V_b$.

The partition sum and correlation functions of the minimal models (2.1) coupled to gravity were computed using matrix models. We start this section by briefly reviewing the analysis of the leading non-perturbative effects in these models due to [6]. In the next subsection we will discuss the D-instantons that give rise to these effects.

JHEP09(2003)057

2.1 Matrix model results

We start with the simplest case, $q = 2$, corresponding to pure gravity. The partition sum of the model is given by the solution of the Painleve-I equation for the second derivative of the partition function $F(\mu)$, $u(\mu) = -\partial_\mu^2 F(\mu)$:

$$u^2(\mu) - \frac{1}{6}u''(\mu) = \mu \,. \tag{2.8}$$

We denoted the cosmological constant in the matrix model by μ; it differs from the Liouville cosmological constant in (2.3), μ_L, by a multiplicative factor.

String perturbation theory is in this case an expansion in even powers of $g_s = \mu^{-5/4}$:

$$u(\mu) = \mu^{1/2} \sum_{h=0}^{\infty} c_h \mu^{-5h/2} \,, \tag{2.9}$$

where $c_0 = 1$, $c_1 = -1/48$, $\dots$, and $c_h \underset{h\to\infty}{\sim} - a^{2h}\Gamma(2h-1/2)$, with $a = 5/8\sqrt{3}$. The series (2.9) is asymptotic, and hence non-perturbatively ambiguous. The size of the leading non-perturbative ambiguities can be estimated as follows. Suppose u and $\tilde{u}$ are two solutions of (2.8) which share the asymptotic behavior (2.9). Then, the difference between them, $\varepsilon = \tilde{u} - u$, is exponentially small in the limit $\mu \to \infty$, and we can treat it perturbatively. Plugging $\tilde{u} = u + \varepsilon$ into (2.8), and expanding to first order in ε, we find that

$$\varepsilon'' = 12u\varepsilon \tag{2.10}$$

which can be written for large μ as

$$\frac{\varepsilon'}{\varepsilon} = r\sqrt{u} + b\frac{u'}{u} + \cdots \tag{2.11}$$

with $r = -2\sqrt{3}$, $b = -1/4$. Using (2.9), $u = \sqrt{\mu} + \cdots$, one finds that

$$\varepsilon \propto \mu^{-\frac{1}{8}} e^{-\frac{8\sqrt{3}}{5}\mu^{\frac{5}{4}}} \,. \tag{2.12}$$

The constant of proportionality in (2.12) is a free parameter of the solution, and cannot be determined solely from the string equation (2.8) without further physical input.

The authors of [6] generalized the analysis above to the case of $(p, p+1)$ minimal models (2.1) coupled to gravity. They found it convenient to parameterize the results in the same way as in the $c = 0$ case, (2.11), i.e. to define the quantity r,

$$\frac{\varepsilon'}{\varepsilon} = r\sqrt{u} + \cdots \,, \tag{2.13}$$

where ε is again the leading non-perturbative ambiguity in $u = -F''$. They found that for general p there is in fact a whole sequence of different solutions for r labeled by two integers (m, n) which vary over the same range as the Kac indices in eq. (2.2). The result for $r_{m,n}$ was found to be:

$$r_{m,n} = -4\sin\frac{\pi m}{p}\sin\frac{\pi n}{p+1} \,. \tag{2.14}$$

Our main purpose in the rest of this section is to reproduce the result (2.14) from Liouville theory.

2.2 Liouville analysis

As discussed in the introduction, general considerations suggest that the leading non-perturbative effects in string theory should be due to contributions of world sheets with holes, with boundary conditions corresponding to localized D-branes. The first question that we need to address is which D-branes should be considered for this analysis.

The minimal model part of the background can be thought of as a finite collection of points. All D-branes corresponding to it are localized and therefore should contribute to the non-perturbative effects. Minimal model D-branes are well understood [18]. They are in one to one correspondence with primaries of the Virasoro algebra (2.2). For our purposes, the main property that will be important is the disk partition sum (or boundary entropy) corresponding to the (m,n) brane, which is given by

$$Z_{m,n} = \left(\frac{8}{p(p+1)}\right)^{1/4} \frac{\sin\frac{\pi m}{p}\sin\frac{\pi n}{p+1}}{\left(\sin\frac{\pi}{p}\sin\frac{\pi}{p+1}\right)^{1/2}}\,. \tag{2.15}$$

What about the Liouville part of the background? The authors of [11] introduced an infinite sequence of localized D-branes, labeled by two integers (m',n'). Which of these branes should we take in evaluating instanton effects?

The analysis of [11] shows that open strings stretched between the (m',n') and (m'',n'') Liouville branes belong to one of a finite number of degenerate representations of the Virasoro algebra with central charge (2.4). The precise set of degenerate representations that arises depends on m',n',m'',n''. Degenerate representations at $c>25$ occur at negative values of world sheet scaling dimension, except for the simplest degenerate operator, 1, whose dimension is zero. One finds [11] that in all sectors of open strings, except those corresponding to $m'=n'=m''=n''=1$ there are negative dimension operators. It is thus natural to conjecture that the only stable D-instantons correspond to the case $(m',n')=(1,1)$, and we will assume this in the analysis below.

To recapitulate, the D-instantons that give rise to the non-perturbative effects in $c<1$ minimal models coupled to gravity have the form: $(1,1)$ brane in Liouville $\times$ (m,n) brane in the minimal model. We next show that these D-branes give rise to the correct leading non-perturbative effects (2.14).

In order to compare to the matrix model results we should in principle use eq. (1.2), and evaluate the disk partition sum of the Liouville $\times$ minimal model D-brane. It turns out, however, much more convenient to compare directly the quantity r (2.13) which appears naturally in the matrix model analysis. The basic point is that this quantity is a natural object to consider in the continuum approach as well.

Indeed, from the continuum point of view, r is the ratio

$$r = \frac{\frac{\partial}{\partial\mu}\log\varepsilon}{\sqrt{u}} = \frac{\frac{\partial}{\partial\mu_L}Z_{\rm disc}}{\sqrt{-\partial^2_{\mu_L}F}}\,, \tag{2.16}$$

where in the numerator we used the fact that $\log\varepsilon$ is the disk partition sum corresponding to the D-instanton (see (1.2)). Thus we see that r is the ratio between the one point function of the cosmological constant operator V_b on the disk, and the square root of its two point function on the sphere. This is a very natural object to consider since it is known in general in CFT that n point functions on the disk behave like the square roots of $2n$ point functions on the sphere. In particular, for the purpose of computing r, we do not have to worry about the multiplicative factor relating μ and μ_L, as it drops out in the ratio; r is a pure number. We next compute it using the results of [11].

We start with the numerator in (2.16). We have

$$\frac{\partial}{\partial \mu_L} Z_{\text{disk}} = Z_{m,n} \times \langle V_b \rangle_{(1,1)}\,, \tag{2.17}$$

where we used the fact that the contribution of the minimal model is simply the disk partition sum (2.15), and the second factor is the one point function of the cosmological constant operator (2.3) on the disk with the boundary conditions corresponding to the (1, 1) D-brane of [11].

$Z_{m,n}$ is given by eq. (2.15). The one point function of V_b can be computed as follows.[2] The annulus partition sum corresponding to open strings ending on the (1, 1) brane of [11] is given in the open string channel by

$$Z_{1,1}(t) = \frac{q^{-\frac{Q^2}{4}}(1-q)}{\eta(q)}\,, \tag{2.18}$$

where $q = \exp(-2\pi t)$ is the modulus of the annulus, and $\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function. Performing the modular transformation to the closed string channel, $t' = 1/t$, one finds [11]:

$$Z_{1,1}(t') = \int_{-\infty}^{\infty} dP \Psi_{1,1}(P)\Psi_{1,1}(-P)\chi_P(q')\,, \tag{2.19}$$

where $\Psi_{1,1}(P)$ is given in appendix A, $q' = \exp(-2\pi t') = \exp(-2\pi/t)$, and

$$\chi_P(q) = \frac{q^{P^2}}{\eta(q)} \tag{2.20}$$

is a non-degenerate Virasoro character. $\Psi_{1,1}(P)$ can be interpreted as an overlap between the (1, 1) boundary state, $B_{1,1}$, and the state with Liouville momentum P,

$$\Psi_{1,1}(P) = \langle B_{1,1}|P\rangle\,. \tag{2.21}$$

Therefore, $\Psi_{1,1}(P)$ is proportional to the one point function on the disk, with (1, 1) boundary conditions, of the Liouville operator V_α, with

$$\alpha = \frac{Q}{2} + iP\,. \tag{2.22}$$

The proportionality constant is a pure number (independent of P and Q). We will not attempt to calculate this number precisely here, since we can deduce it by matching any

[2]We are grateful to Al. Zamolodchikov for very useful discussions of this issue.

one of the matrix model predictions, and then use it in all other calculations, but we will next mention a few contributions to it, for illustrative purposes.

First, in (2.19) the P integral runs over the whole real line, but $|P\rangle$ and $|-P\rangle$ are the same state, due to reflection from the Liouville wall, so we should replace $\int_{-\infty}^{\infty} dP = 2\int_0^{\infty} dP$. Hence, the physical wavefunctions which should appear in (2.21) are $\sqrt{2}\Psi_{1,1}$. Furthermore, the state $|P\rangle$ in (2.21) is normalized as $\langle P|P'\rangle = \delta(P-P')$, for consistency with (2.19). On the other hand, the state created by acting with the Liouville exponential V_α (2.7), with α given by (2.22), is normalized to $\pi\delta(P-P')$. Finally, when computing $\langle V_\alpha(0)\rangle_{\rm disk}$, the part of the $\mathrm{SL}(2,\mathbb{R})$ symmetry of the disk related to rotations around the origin is unfixed. This gives an extra factor of $1/2\pi$ in the one point function. Altogether, one finds

$$\langle V_\alpha(0)\rangle_{(1,1)} = C\cdot\sqrt{2}\cdot\sqrt{\pi}\cdot\frac{1}{2\pi}\cdot\Psi_{1,1} \tag{2.23}$$

where α is given by (2.22) and C is an undetermined constant which we will fix by comparing to matrix model results (we will find that $C=2$).

We can now plug in (2.23) into (2.17) and, using the form of $\Psi_{1,1}$ given in (A.3), find

$$\langle V_b(0)\rangle_{(1,1)} = -C\,\frac{2^{1/4}\sqrt{\pi}[\pi\mu_L\gamma(b^2)]^{\frac{1}{2}(1/b^2-1)}}{b\Gamma(1-b^2)\Gamma(1/b^2)}\,. \tag{2.24}$$

We next move on to the denominator of (2.16). This is given by the two point function $\langle V_bV_b\rangle_{\rm sphere}$. This quantity was calculated in [19, 20]. It is convenient to first compute the three point function $\langle V_bV_bV_b\rangle_{\rm sphere}$ and then integrate once, to avoid certain subtle questions regarding the fixing of the $\mathrm{SL}(2,\mathbb{C})$ Conformal Killing Group of the sphere. One has (see appendix A):

$$\langle V_bV_bV_b\rangle_{\rm sphere} = b^{-1}\left[\pi\mu_L\right]^{1/b^2-2}\left[\gamma(b^2)\right]^{1/b^2}\gamma\left(2-\frac{1}{b^2}\right). \tag{2.25}$$

Integrating once w.r.t. $-\mu_L$ we find

$$\langle V_bV_b\rangle_{\rm sphere} = \frac{1/b^2-1}{\pi b}\left[\pi\mu_L\gamma(b^2)\right]^{1/b^2-1}\gamma(b^2)\gamma\left(1-\frac{1}{b^2}\right). \tag{2.26}$$

We are now ready to compute r, (2.16). Plugging in (2.15), (2.24) and (2.26) into (2.16), we find

$$r_{m,n} = -2C\sin\frac{\pi m}{p}\sin\frac{\pi n}{p+1}\,, \tag{2.27}$$

which agrees with the matrix model result (2.14) if we set $C=2$. Since C is independent of m, n and p, we can fix it by matching to any one case, and then use it in all others. We conclude that the Liouville analysis gives the same results as the matrix model one.

3. Non-perturbative effects in compactified $c = 1$ string theory

In this section we discuss the leading non-perturbative effects in (euclidean) two dimensional string theory. The world sheet description contains a field x, compactified on a circle of radius R, $x \simeq x + 2\pi R$, coupled to gravity. The conformal gauge lagrangian describing this system is

$$\mathcal{L} = \frac{1}{4\pi}\left[(\partial x)^2 + (\partial\phi)^2 + 2\hat{R}\phi + \mu_{L}\phi e^{2\phi} + \lambda e^{(2-R)\phi}\cos[R(x_L - x_R)]\right]. \qquad (3.1)$$

where we chose $\alpha' = 1$, and perturbed the $c = 1$ model by a Sine-Gordon type interaction, $\lambda\cos[R(x_L - x_R)]$, which gives rise to the last term in (3.1) after coupling to gravity. The perturbation is relevant for $R < 2$, and we will restrict to that case in the discussion below.

We will next describe the matrix model predictions for the leading non-perturbative effects corresponding to (3.1), and then verify some of them in Liouville theory.

3.1 Matrix model analysis

The model (3.1) was solved using matrix model techniques in [16], extending some earlier results [21]–[25]. In particular, it was shown in [16] that the Legendre transform $\mathcal{F}$ of the string partition sum F satisfies the Toda differential equation:

$$\frac{1}{4}\lambda^{-1}\partial_\lambda\lambda\partial_\lambda\mathcal{F}(\mu,\lambda) + \exp\left[-4\sin^2\left(\frac{1}{2}\frac{\partial}{\partial\mu}\right)\mathcal{F}(\mu,\lambda)\right] = 1\,. \qquad (3.2)$$

The initial condition for this equation is supplied by the unperturbed $c = 1$ string theory on a circle

$$\begin{aligned}\mathcal{F}(\mu,0) &= \frac{R}{4}\,\mathrm{Re}\int_{\Lambda^{-1}}^{\infty}\frac{ds}{s}\frac{e^{-i\mu s}}{\sinh\frac{s}{2}\sinh\frac{s}{2R}} \qquad (3.3)\\ &= -\frac{R}{2}\mu^2\log\frac{\mu}{\Lambda} - \frac{1}{24}\left(R + \frac{1}{R}\right)\log\frac{\mu}{\Lambda} + R\sum_{h=2}^{\infty}\mu^{2-2h}c_h(R) + O(e^{-2\pi\mu}) + O(e^{-2\pi R\mu})\,,\end{aligned}$$

where the genus h term $c_h(R)$ is a known polynomial in $1/R$. The function $\mathcal{F}(\lambda,\mu)$ has a genus expansion which follows from (3.2). The string partition sum F is obtained from $\mathcal{F}$ by flipping the sign of the genus zero term [26].

We will need below some features of the solution of equations (3.2), (3.3), which we review next. In order to study the solution of (3.2), (3.3), it is convenient to introduce the parameters

$$y = \mu\xi\,, \qquad \xi = (\lambda\sqrt{R-1})^{-\frac{2}{2-R}}\,. \qquad (3.4)$$

This parameterization is useful when λ is large, so that the last term in (3.1) sets the scale, and one can study the cosmological term perturbatively in μ. As explained in [16], this can only be done for $R > 1$. This is the physical origin of the branch cut in the definition of ξ in (3.4). In the regime where λ sets the scale, the genus expansion of the string partition sum is an expansion in even powers of ξ:

$$\mathcal{F}(\mu,\lambda) = \lambda^2 + \xi^{-2}\left[\frac{R}{2}y^2\log\xi + f_0(y)\right] + \left[\frac{R+R^{-1}}{24}\log\xi + f_1(y)\right] + \sum_{h=2}^{\infty}\xi^{2h-2}f_h(y)\,. \quad (3.5)$$

The partition sum on the sphere, $\mathcal{F}_0(\mu,\lambda)$, is given by the second term on the r.h.s. of (3.5). One can show [16] that at this order, the full Toda equation (3.2) reduces to an algebraic equation for $X = \partial_y^2 f_0$

$$y = e^{-\frac{1}{R}X} - e^{-\frac{R-1}{R}X}\,. \qquad (3.6)$$

This leads to the following perturbative expansion of $\mathcal{F}_0(\mu,\lambda)$ in powers of λ:

$$\mathcal{F}_0(\mu,\lambda) = \frac{R}{2}\mu^2 \log\frac{\Lambda}{\mu} + R\mu^2 \sum_{n=1}^{\infty} \frac{1}{n!}\left((1-R)\mu^{R-2}\lambda^2\right)^n \frac{\Gamma(n(2-R)-2)}{\Gamma(n(1-R)+1)}, \tag{3.7}$$

which agrees with the original (conjectured) result for this quantity in [21]. The genus $h \geq 1$ terms in the expansion (3.5) can be computed in principle by substituting (3.5) into (3.2), but only f_0 and f_1 are known in closed form. Nevertheless, we will see below that the large h behavior of f_h can be obtained directly from the differential equation (3.2), by computing the leading non-perturbative corrections to (3.5). We next turn to these corrections.

At $\lambda = 0$, there are two types of non-perturbative corrections to the $1/\mu$ expansion (3.3), associated with the poles of the integrand in that equation. These occur at $s = 2\pi ik$ and $s = 2\pi Rik$, $k \in \mathbb{Z}$, and give rise to the non-perturbative effects indicated on the second line of (3.3), $\exp(-2\pi\mu k)$ and $\exp(-2\pi R\mu k)$, respectively.

At finite λ, the situation is more interesting. The series of non-perturbative corrections

$$\Delta\mathcal{F} = \sum_{n=1}^{\infty} C_n e^{-2\pi n\mu} \tag{3.8}$$

gives rise to an exact solution of the *full* equation (3.2) [16]. Thus, the corresponding instantons are insensitive to the presence of the sine-Gordon perturbation! We will return to this interesting fact, and explain its interpretation in Liouville theory, in the next subsection.

The second type of corrections, which starts at $\lambda = 0$ like $\Delta\mathcal{F} = e^{-2\pi R\mu}$, does not solve the full equation (3.2), and gets λ dependent corrections. To study these corrections one can proceed in a similar way to that described in section 2.1.

Let $\mathcal{F}$ and $\tilde{\mathcal{F}} = \mathcal{F} + \varepsilon$ be two solutions of the Toda equation (3.2). The linearized equation for ε reads

$$\frac{1}{4}\lambda^{-1}\partial_\lambda \lambda \partial_\lambda \varepsilon(\mu,\lambda) - 4e^{-\partial_\mu^2 \mathcal{F}_0(\mu,\lambda)} \sin^2\left(\frac{1}{2}\frac{\partial}{\partial\mu}\right)\varepsilon(\mu,\lambda) = 0, \tag{3.9}$$

where in the exponential in the second term we approximated

$$4\sin^2\left(\frac{1}{2}\frac{\partial}{\partial\mu}\right)\mathcal{F}(\mu,\lambda) \simeq \partial_\mu^2 \mathcal{F}_0 = R\log\xi + X(y). \tag{3.10}$$

This is similar to the fact that in the discussion of section 2, one can replace u in eq. (2.10) by its limit as $\mu \to \infty$. After the change of variables from (λ,μ) to (y,μ), equation (3.9) can be written as

$$\alpha\frac{y^2}{\mu^2}(y\partial_y)^2\varepsilon(\mu,y) - 4e^{-X(y)}\sin^2\left(\frac{1}{2}\left(\frac{\partial}{\partial\mu} + \frac{y}{\mu}\frac{\partial}{\partial y}\right)\right)\varepsilon(\mu,y) = 0, \tag{3.11}$$

where $\alpha \equiv \frac{R-1}{(2-R)^2}$.

Since we know that ε is non-perturbatively small in the $\mu \to \infty$ limit, we use the following ansatz for it:

$$\varepsilon(\mu,y) = P(\mu,y)e^{-\mu f(y)}. \tag{3.12}$$

Here P is a power-like prefactor in g_s, and $f(y)$ is the function we are interested in (the analogue of r in the minimal models of section 2). Substituting (3.12) into (3.11) and keeping only the leading terms in the $\mu \to \infty$ limit, one finds the following first order differential equation:

$$\sqrt{\alpha}\, e^{\frac{1}{2}X(y)}(1 - y\partial_y)g(y) = \pm \sin\left[\partial_y g(y)\right] , \tag{3.13}$$

where we introduced

$$g(y) = \frac{1}{2} y f(y) . \tag{3.14}$$

The $\pm$ in (3.13) is due to the fact that one in fact finds the square of this equation.

Equation (3.13) is a first order differential equation in y, and to solve it we need to specify boundary conditions. As discussed earlier for the perturbative series, it is natural to specify these boundary conditions at $\lambda \to 0$, or $y \to \infty$ (see (3.4)). We saw in the discussion of (3.3) that there are two solutions, $f(y \to \infty) \to 2\pi$ or $2\pi R$. This implies via (3.14) that $g(y \to \infty) \simeq \pi y$ or $\pi R y$. We already saw that $g(y) = \pi y$ gives an exact solution of (3.2), and this is true for (3.13) as well (as it should be). Thus, to study non-trivial non-perturbative effects, we must take the other boundary condition,

$$g(y \to \infty) \simeq \pi R y . \tag{3.15}$$

Interestingly, eq. (3.13) is exactly solvable. We outline the solution in appendix B. For the initial condition (3.15), the solution can be written as

$$g(y) = y\phi(y) \pm \frac{1}{\sqrt{\alpha}} e^{-\frac{1}{2}X(y)} \sin\phi(y) , \tag{3.16}$$

where $\phi(y) = \partial_y g$ satisfies the equation

$$e^{\frac{2-R}{2R}X(y)} = \pm\sqrt{R-1}\, \frac{\sin\left(\frac{1}{R}\phi\right)}{\sin\left(\frac{R-1}{R}\phi\right)} . \tag{3.17}$$

The solution with the minus sign in (3.17) can be shown to be unphysical (see below). We will thus use the solution with a plus sign.

Equations (3.16), (3.17) are the main result of this subsection. We next discuss some features of the corresponding non-perturbative effects.

Consider first the situation for small λ, or large y. The first three terms in the expansion of $\phi(y)$ are

$$\phi(y) \approx \pi R \pm \frac{R\sin(\pi R)}{\sqrt{R-1}}\, y^{-\frac{2-R}{2}} + \frac{R}{2}\sin(2\pi R)\, y^{-(2-R)} . \tag{3.18}$$

This gives the following result for $f(y)$ (3.12):

$$\begin{aligned} f(y) &= 2\pi R \pm \frac{4\sin(\pi R)}{\sqrt{R-1}}\, y^{-\frac{2-R}{2}} + \frac{R\sin(2\pi R)}{R-1} y^{-(2-R)} + O(y^{-3(2-R)/2}) \\ &= 2\pi R \pm 4\sin(\pi R)\, \mu^{-\frac{2-R}{2}}\, \lambda + R\sin(2\pi R)\, \mu^{-(2-R)}\, \lambda^2 + O(\lambda^3) . \end{aligned} \tag{3.19}$$

We see that for large y, the expansion parameter is $y^{-\frac{(2-R)}{2}} \sim \lambda$, as one would expect.

Another interesting limit is $\mu \to 0$ at fixed λ, i.e. $y \to 0$, which leads to the Sine-Liouville model, (3.1) with $\mu_L = 0$. For $R = 3/2$, this model is equivalent to the euclidean black hole background [27, 16]. As we see in (3.6), in this limit $X \to 0$. The first two terms in the expansion of ϕ around this point are

$$\phi(y) = \phi_0 + \frac{R}{2}\left((R-1)\cot\left(\frac{R-1}{R}\phi_0\right) - \cot\left(\frac{1}{R}\phi_0\right)\right)^{-1} y + O(y^2)\,, \tag{3.20}$$

where ϕ_0 is defined by the equation

$$\frac{\sin\left(\frac{1}{R}\phi_0\right)}{\sin\left(\frac{R-1}{R}\phi_0\right)} = \pm\frac{1}{\sqrt{R-1}}\,. \tag{3.21}$$

The function $f(y)$ is given in this limit by the expansion

$$f(y) = \pm\frac{2(2-R)}{y\sqrt{R-1}}\sin\phi_0 + 2\phi_0 + O(y)\,. \tag{3.22}$$

Note that the behavior of f as $y \to 0$, $f \sim 1/y$, leads to a smooth limit as $\mu \to 0$ at fixed λ. The non-perturbative effect (3.12) goes like $\exp(-\mu f(y))$, so that as $y \to 0$ the argument of the exponential goes like $\mu/y = 1/\xi$, and all dependence on μ disappears.

For $R = 3/2$, which corresponds to the euclidean black hole, the equations simplify. One can explicitly find ϕ_0 because (3.21) gives

$$\cos\frac{\phi_0}{3} = \pm\frac{1}{\sqrt{2}} \Rightarrow \phi_0 = \frac{3\pi}{4} \text{ or } \phi_0 = \frac{9\pi}{4}\,. \tag{3.23}$$

As a result, one finds at this value of the radius a simple result

$$\mu f(y) = \pm\frac{\mu}{y} + \frac{(6\mp 3)\pi}{2}\mu + \cdots = \pm\frac{1}{2}\lambda^4 + \frac{(6\mp 3)\pi}{2}\mu + \cdots\,. \tag{3.24}$$

Note that the solution with the minus sign leads to a growing exponential, $e^{\frac{1}{2}\lambda^4}$. Therefore, it is not physical, as mentioned above. For the particular case $R = 3/2$ one can write the solution for $f(y)$ more explicitly. The result is given in appendix B.

Another nice consistency check on our solution is the RG flow from $c = 1$ to $c = 0$ CFT coupled to gravity, implicit in (3.1). Before coupling to gravity, the Sine-Gordon model associated to (3.1) describes the following RG flow. In the UV, the Sine-Gordon coupling effectively goes to zero, and one approaches the standard CFT of a compact scalar field. In the IR, the potential given by the Sine-Gordon interaction gives a world sheet mass to x, and the model approaches a trivial $c = 0$ fixed point. As shown in [21, 23], this RG flow manifests itself after coupling to gravity in the dependence of the physics on μ. Large μ corresponds to the UV limit; in it, all correlators approach those of the $c = 1$ theory coupled to gravity. Decreasing μ corresponds in this language to the flow to the IR, with the $c = 0$ behavior recovered as μ approaches a critical value μ_c, which is given by:

$$\mu_c = -(2-R)(R-1)^{\frac{R}{2-R}}\lambda^{\frac{2}{2-R}}\,. \tag{3.25}$$

The non-perturbative contributions to the partition sum computed in this section must follow a similar pattern. In particular, $f(y)$ must exhibit a singularity as $y \to y_c$, with

$$y_c = -(2-R)(R-1)^{\frac{R-1}{2-R}} \tag{3.26}$$

and furthermore, the behavior of f near this singularity should reproduce the non-perturbative effects of the $c=0$ model coupled to gravity discussed in section 2.

The singularity at $y = y_c$ corresponds to a critical point of (3.6), near which the relation between y and X degenerates:

$$\frac{y_c - y}{y_c} \simeq \frac{R-1}{2R^2}(X - X_c)^2 + O\left((X-X_c)^3\right) . \tag{3.27}$$

Solving for the critical point, one finds that

$$e^{-\frac{2-R}{2R}X_c} = \sqrt{R-1} . \tag{3.28}$$

The corresponding y_c is indeed given by (3.26). We can now solve for $f(y)$, near $y = y_c$. Substituting (3.28) in (3.17) we find

$$\frac{\sin\left(\frac{1}{R}\phi\right)}{\sin\left(\frac{R-1}{R}\phi\right)} = \frac{1}{R-1} . \tag{3.29}$$

Thus, the $c=0$ critical point corresponds to $\phi \to 0$.[3] The first two terms in the expansion of ϕ around the singularity are

$$\phi(y) = \sqrt{3}(X_c - X)^{1/2} - \frac{\sqrt{3}(R^2 - 2R + 2)}{20R^2}(X_c - X)^{3/2} + O\left((X_c - X)^{5/2}\right) . \tag{3.30}$$

Substituting this in (3.16) one finds

$$g(y) = -y_c \frac{2\sqrt{3}(R-1)}{5R^2}(X_c - X)^{5/2} + O\left((X_c - X)^{5/2}\right) \tag{3.31}$$

or, using (3.14):

$$f(y) \approx -\frac{8\sqrt{3}}{5}\left(\frac{2R^2}{R-1}\right)^{1/4}\left(\frac{\mu_c - \mu}{\mu_c}\right)^{5/4} . \tag{3.32}$$

The power of $\mu - \mu_c$ is precisely right to describe a leading non-perturbative effect in pure gravity. It is interesting to compare the coefficient in (3.32) to what is expected in pure gravity. It is most convenient to do this by again computing r (2.11). u is computed by evaluating the leading singular term as $\mu \to \mu_c$ in $\partial_\mu^2 \mathcal{F}_0 = R \log \xi + X(y)$. One finds

$$r = -2\sqrt{3}\left(\frac{2R^2}{R-1}\right)^{1/4}\left(\frac{\mu_c - \mu}{\mu_c}\right)^{1/4}(X_c - X)^{-1/2} = -2\sqrt{3} \tag{3.33}$$

in agreement with the result (2.14) for pure gravity. This provides another non-trivial consistency check of our solution.

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[3] Note that if we chose the minus sign in (3.17), we would find a more complicated solution for ϕ. One can show that it would lead to the wrong critical behavior. This is an additional check of the fact that the physical solution corresponds to the plus sign in (3.17).

3.2 Liouville analysis

In this subsection we will discuss the Liouville interpretation of the matrix model results presented in the previous subsection. Consider first the unperturbed $c = 1$ theory corresponding to $\lambda = 0$ in (3.1). As we saw, in the matrix model analysis one finds two different types of leading non-perturbative effects (see (3.3)), $\exp(-2\pi\mu)$, and $\exp(-2\pi R\mu)$. It is not difficult to guess the origin of these non-perturbative effects from the Liouville point of view. The $\exp(-2\pi\mu)$ contribution is due to a D-brane that corresponds to a $(1,1)$ Liouville brane $\times$ a Dirichlet brane in the $c = 1$ CFT, i.e. a brane located at a point on the circle parameterized by x. Similarly, the $\exp(-2\pi R\mu)$ term comes from a brane wrapped around the x circle.

This identification can be verified in the same way as we did for minimal models in section 2. To avoid normalization issues, one can again calculate the quantity r (2.16). The matrix model prediction for the Neumann brane[4] is

$$r = -\frac{2\pi\sqrt{R}}{\sqrt{\log\frac{\Lambda}{\mu}}}\,, \tag{3.34}$$

where we used the sphere partition function $\mathcal{F}_0(\mu, 0)$ of the unperturbed compactified $c = 1$ string given by the first term in (3.7).

The CFT calculation is similar to that performed in the $c < 1$ case. The partition function of $c = 1$ CFT on a disk with Neumann boundary conditions is well known; the calculation is reviewed in appendix C. One finds

$$Z_{\mathrm{Neumann}} = 2^{-1/4}\sqrt{R}\,. \tag{3.35}$$

The disk amplitude corresponding to the $(1,1)$ Liouville brane, and the two-point function on the sphere are computed using equations (2.24) and (2.26), in the limit $b \to 1$. The limit is actually singular, but computing everything for generic b and taking the limit at the end of the calculation leads to sensible, finite results. The leading behavior of (2.24) as $b \to 1$ is

$$\langle V_b(0)\rangle_{(1,1)} \approx -\frac{2^{5/4}\sqrt{\pi}}{\Gamma(1-b^2)}\,, \tag{3.36}$$

with the constant C in (2.24) chosen to be equal to 2, as in the minimal model analysis of equations (2.23) – (2.27). The two point function (2.26) approaches

$$\partial^2_{\mu_L}\mathcal{F}_0 \simeq -\frac{\log\mu_L}{\pi\Gamma^2(1-b^2)}\,. \tag{3.37}$$

Substituting (3.35), (3.36) and (3.37) into (2.16) gives the result (3.34). This provides a non-trivial check of the statement in section 2, that the constant C in eq. (2.23) is a pure number, independent of all the parameters of the model.

The agreement of (3.34) with the Liouville analysis supports the identification of the Neumann D-branes as the source of the non-perturbative effects $\exp(-2\pi R\mu)$. A similar

[4] Similar formulae can be written for the Dirichlet brane.

JHEP09(2003)057

analysis leads to the same conclusion regarding the relation between the Dirichlet $c = 1$ branes and the non-perturbative effects $\exp(-2\pi\mu)$ (the two kinds of branes are related by T-duality).

Having understood the structure of the unperturbed theory, we next turn to the theory with generic λ. In the matrix model we found that the non-perturbative effects associated with the Dirichlet brane localized on the x circle are in fact independent of λ (see (3.8) and the subsequent discussion). How can we understand this statement from the Liouville point of view?

The statement that (3.8) is an exact solution of the Toda equation (3.2) corresponds in the continuum formulation to the claim that the disk partition sum of the model (3.1), with $(1,1)$ boundary conditions for Liouville, and Dirichlet boundary conditions for the matter field x is independent of λ. In other words, all n-point functions of the operator given by the last term in (3.1) on the disk vanish

$$\left\langle \left(\int d^2 z e^{(2-R)\phi} \cos[R(x_L - x_R)] \right)^n \right\rangle_{(1,1)\times\text{Dirichlet}} = 0\,. \tag{3.38}$$

Is it reasonable to expect (3.38) to be valid from the world sheet point of view? For odd n (3.38) is trivially zero because of winding number conservation. Indeed, the Dirichlet boundary state for x breaks translation invariance, but preserves winding number. The perturbation in (3.1) carries winding number, and for odd n all terms in (3.38) have non-zero winding number. Thus, the correlator vanishes.

For even n one has to work harder, but it is still reasonable to expect the amplitude to vanish in this case. Indeed, consider the T-dual statement to (3.38), that the n point functions of the momentum mode $\cos(x/R)$, on the disk with $(1,1) \times$ Neumann boundary conditions, vanish. This is reasonable since the operator whose correlation functions are being computed localizes x at the minima of the cosine, while the D-brane on which the string ends is smeared over the whole circle. It might be possible to make this argument precise by using the fact that in this case the D-instanton preserves a different symmetry from that preserved by the perturbed theory, and thus it should not contribute to the non-perturbative effects.

To summarize, the matrix model analysis predicts that (3.38) is valid. We will not attempt to prove this assertion here from the Liouville point of view (it would be nice to verify it even for the simplest case, $n = 2$), and instead move on to discuss the non-perturbative effects due to the localized branes on the x circle.

The non-trivial solution of eq. (3.13) given by (3.16), (3.17) should correspond from the Liouville point of view to the disk partition sum of the $(1,1)$ Liouville brane which is wrapped around the x circle. The prediction is that

$$\left\langle \left(\int d^2 z e^{(2-R)\phi} \cos[R(x_L - x_R)] \right)^n \right\rangle_{(1,1)\times\text{Neumann}} \tag{3.39}$$

are the coefficients in the expansion of $f(y)$ (3.12) in a power series in λ, the first terms of which are given by (3.19). It would be very nice to verify this prediction directly using

JHEP09(2003)057

Liouville theory, but in general this seems hard given the present state of the art. A simple check that can be performed using results in [11] is to compare the order λ term in (3.19) with the $n = 1$ correlator (3.39). We next compare the two.

Like in the other cases studied earlier, it is convenient to define a dimensionless quantity, ρ, given by the ratio of the one point function on the disk (3.39) and the square root of the appropriate two point function on the sphere,

$$\rho = \left. \frac{\frac{\partial}{\partial\lambda} \log \varepsilon}{\sqrt{-\partial_\lambda^2 \mathcal{F}_0}} \right|_{\lambda=0} . \tag{3.40}$$

The matrix model result (3.19), (3.7), for this quantity is

$$\rho = - \left. \frac{\mu \frac{\partial}{\partial\lambda} f}{\sqrt{-\partial_\lambda^2 \mathcal{F}_0}} \right|_{\lambda=0} = -2\sqrt{2} \sin(\pi R) . \tag{3.41}$$

In the Liouville description, ρ is given by

$$\rho = \frac{B_{\mathcal{T}} \langle V_{b-\frac{R}{2}} \rangle_{(1,1)}}{\sqrt{-\langle \mathcal{T}^2 \rangle}} . \tag{3.42}$$

Here, $\mathcal{T} = \cos R(x_L - x_R) V_{b-\frac{R}{2}}$. $B_{\mathcal{T}}$ is the one point function of $\cos R(x_L - x_R)$ on the disk. It has the same value as (3.35) (see appendix C)

$$B_{\mathcal{T}} = 2^{-1/4} \sqrt{R} . \tag{3.43}$$

The one point function of the operator $V_{b-\frac{R}{2}}$ is related to the wavefunction (A.3) with momentum $iP = b - R/2 - Q/2$ and is given by

$$\langle V_{b-\frac{R}{2}}(0) \rangle_{(1,1)} = - \frac{2^{5/4} \sqrt{\pi} [\pi \mu_L \gamma(b^2)]^{\frac{1}{2}(1/b^2 - 1 + R/b)}}{b \Gamma(1 - b^2 + Rb) \Gamma(1/b^2 + R/b)} . \tag{3.44}$$

The two-point function of $\mathcal{T}$ on the sphere is computed as above from the three point function (A.10). One finds

$$\langle \mathcal{T}^2 \rangle = \frac{\left(\frac{1}{b^2} + \frac{R}{b} - 1\right)}{2\pi b} \left[\pi \mu_L \gamma(b^2)\right]^{\frac{1}{b^2} + \frac{R}{b} - 1} \gamma\left(b^2 - Rb\right) \gamma\left(1 - \frac{1}{b^2} - \frac{R}{b}\right) . \tag{3.45}$$

Substituting these results into (3.42) leads, in the limit $b = 1$, to (3.41). We see that the Liouville results are again in complete agreement with the corresponding matrix model calculation.

4. Discussion

In this paper we studied non-perturbative effects in $c \leq 1$ non-critical string theory, using both matrix model and continuum (Liouville) methods. We showed that the matrix model results correspond in the continuum approach to D-instanton effects due to localized branes in Liouville theory, the $(1,1)$ branes constructed in [11]. Our work is motivated by the recent proposal of [13], but the details are different.

For $c < 1$ unitary minimal models, we showed that the contribution of these branes reproduces the matrix model results of [6]. For $c = 1$ we used the results of [16] to derive the leading non-perturbative effects in the matrix model description of the Sine-Gordon model coupled to gravity, as a function of the compactification radius and the Sine-Gordon coupling, and reproduced some of these predictions using the Liouville branes of [11].

The matrix model analysis led in addition to a number of predictions regarding the properties of D-instantons in $c = 1$ string theory. In particular, we showed that an infinite number of correlation functions given in eq. (3.38) must vanish, and the correlation functions (3.39) are given by the expansion of $f(y)$ (3.14) in a power series in λ, the first few terms of which appear in (3.19).

Another interesting set of matrix model predictions concerns the limit $\mu \to 0$, in which one finds the Sine-Liouville model, (3.1) with $\mu_L = 0$. The expansion of $f(y)$ around $y = 0$, (3.22), provides information about the disk partition sum and correlation functions of closed string operators in the presence of D-branes in this model. For the particular case $R = 3/2$, these predictions apply also to the $\mathrm{SL}(2)/\mathrm{U}(1)$ (cigar) background corresponding to a euclidean two dimensional black hole. In the cigar, the D-branes in question are wrapped around the angular direction of the cigar, and are localized near the tip. Our results give rise to predictions for correlation functions on the disk with these boundary conditions.

Our analysis also resolves a previously open problem, of describing world sheet gravity on the disk with the boundary conditions described in [11] using the matrix model approach. It opens the possibility of studying gravity on AdS_2 using matrix model techniques.

An interesting open problem is how to go beyond the calculation of the leading non-perturbative contributions to the amplitudes (1.1), and study the perturbative expansion around the D-instantons discussed in this paper. In the matrix model approach one finds that the leading non-perturbative terms behave like

$$\mathcal{A}_{1-inst.} \sim C \; g_s^{f_A} \; e^{-\frac{f_D}{g_s}} \; , \tag{4.1}$$

where f_D was discussed in this paper, and f_A is known as well.[5] The constant C is ambiguous in the matrix model.

In the continuum Liouville description, f_D has been computed in this paper, and it would be interesting to compute f_A and understand whether C is ambiguous or can be determined. In order to study these issues, one has to understand the perturbative expansion about the D-instantons discussed in this paper. In particular, it is natural to expect that f_A will arise from the annulus with boundary conditions corresponding to the $(1,1)$ brane (this gives the correct scaling with g_s).

The discussion of this paper has some interesting higher dimensional generalizations. For example, if one replaces Liouville by $N = 2$ Liouville, and the minimal models by

[5]For the partition sum one finds a simple universal behaviour. For the minimal models, in all cases that we are aware of, $F_{\text{non-pert}} \sim g_s^{1/2} e^{-\frac{f_D}{g_s}}$ (see [28]). For the $c = 1$ model, equation (3.8) suggests that $\mathcal{F}_{\text{non-pert}} \sim g_s^0 e^{-\frac{f_D}{g_s}}$.

JHEP09(2003)057

$N = 2$ minimal models, one finds the background

$$\frac{\mathrm{SL}(2)_k}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(2)_k}{\mathrm{U}(1)} \tag{4.2}$$

describing $NS5$-branes spread out on a circle [29, 30]. The D-instantons analogous to those studied in this paper correspond to euclidean D0-branes stretched between pairs of fivebranes. From the point of view of the geometry (4.2), they are described by D-branes living near the tip of the cigar, which are $N = 2$ superconformal generalizations of the $(1, 1)$ brane of [11]. In particular, the spectrum of open strings ending on them corresponds to the $N = 2$ superconformal block of the identity.

Like in our case, the D-instantons are obtained by tensoring the above $N = 2$ Liouville branes with the different D-branes in the $N = 2$ minimal model. Understanding the non-perturbative effects due to these instantons is an interesting open problem. Studying the branes discussed in this paper in more detail is a very useful warmup exercise for the fivebrane problem. It might suggest hints for finding an analog of the matrix model for fivebranes.

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A. Correlation functions in Liouville theory

In this appendix we summarize some results on Liouville field theory, from [11, 19, 20], which are used in the text. The one-point function of the operator (2.7) in Liouville theory on the disk with boundary conditions corresponding to the $(1, 1)$ brane is given by

$$U(\alpha) = \frac{[\pi\mu_L\gamma(b^2)]^{-\alpha/b} Q\Gamma(bQ)\Gamma(Q/b)}{(Q - 2\alpha)\Gamma(b(Q - 2\alpha))\Gamma((Q - 2\alpha)/b)}\,, \tag{A.1}$$

where

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1 - x)}\,. \tag{A.2}$$

$U(\alpha)$ is the normalized one-point function, i.e. the one point function divided by the disk partition function. As discussed in the text, the unnormalized one point function of V_α is related to the wavefunction with momentum $iP = \alpha - Q/2$ defined by (2.19), which is given by

$$\Psi_{1,1}(P) = \frac{2^{3/4} 2\pi i P [\pi\mu_L\gamma(b^2)]^{-iP/b}}{\Gamma(1 - 2ibP)\Gamma(1 - 2iP/b)}\,. \tag{A.3}$$

The three-point function on the sphere is (suppressing the standard dependence on the world sheet positions of the vertex operators)

$$\begin{aligned}\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3}\rangle_{\text{sphere}} = & \left[\pi\mu_L \gamma(b^2) b^{2-2b^2}\right]^{(Q-\sum\alpha_i)/b} \times \\ & \times \frac{\Upsilon_0 \Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon(\alpha_1+\alpha_2-\alpha_3)\Upsilon(\alpha_2+\alpha_3-\alpha_1)\Upsilon(\alpha_3+\alpha_1-\alpha_2)},\end{aligned} \tag{A.4}$$

where

$$\Upsilon(x) = \exp\left\{\int_0^\infty \frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2 e^{-t} - \frac{\sinh^2\left(\frac{Q}{2}-x\right)\frac{t}{2}}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right]\right\} \tag{A.5}$$

and

$$\Upsilon_0 \stackrel{\text{def}}{=} \left.\frac{d\Upsilon(x)}{dx}\right|_{x=0} = \Upsilon(b)\,. \tag{A.6}$$

An important property of the function $\Upsilon(x)$ is the set of recursion relations it satisfies,

$$\Upsilon(x+b) = \gamma(bx) b^{1-2bx}\Upsilon(x)\,, \tag{A.7}$$

$$\Upsilon\left(x+\frac{1}{b}\right) = \gamma\left(\frac{x}{b}\right) b^{2x/b-1}\Upsilon(x)\,. \tag{A.8}$$

Special cases of these relations that are used in the text are:

$$\frac{\Upsilon(2b)}{\Upsilon(b)} = \gamma(b^2) b^{1-2b^2}\,, \tag{A.9}$$

$$\frac{\Upsilon(2b)}{\Upsilon\left(2b-\frac{1}{b}\right)} = \gamma(2-1/b^2) b^{3-2/b^2}\,, \tag{A.10}$$

$$\frac{\Upsilon(2b-R)}{\Upsilon(b-R)} = \gamma(b(b-R))\, b^{1-2b^2+2bR}\,, \tag{A.11}$$

$$\frac{\Upsilon(2b-R)}{\Upsilon(2b-1/b-R)} = \gamma\left(2-\frac{1}{b^2}-\frac{R}{b}\right) b^{3-2/b^2-2R/b}\,. \tag{A.12}$$

Using these relations one finds:

$$\begin{aligned}\langle V_b V_b V_b\rangle_{\text{sphere}} &= \left[\pi\mu_L \gamma(b^2) b^{2-2b^2}\right]^{1/b^2-2} \frac{\Upsilon_0}{\Upsilon(2b-1/b)}\left(\frac{\Upsilon(2b)}{\Upsilon(b)}\right)^3 && \text{(A.13)}\\ &= b^{-1}\left[\pi\mu_L\right]^{1/b^2-2}\left[\gamma(b^2)\right]^{1/b^2}\gamma\left(2-\frac{1}{b^2}\right), && \text{(A.14)}\\ \langle V_{b-\frac{R}{2}} V_{b-\frac{R}{2}} V_b\rangle_{\text{sphere}} &= \left[\pi\mu_L \gamma(b^2) b^{2-2b^2}\right]^{\frac{1}{b^2}-2+\frac{R}{b}} \frac{\Upsilon_0\Upsilon(2b)\Upsilon^2(2b-R)}{\Upsilon(2b-1/b-R)\,\Upsilon(b-R)\Upsilon^2(b)} && \text{(A.15)}\\ &= b^{-1}\left[\pi\mu_L\gamma(b^2)\right]^{\frac{1}{b^2}+\frac{R}{b}-1}\gamma(b^2)\gamma\left(b(b-R)\right)\gamma\left(2-\frac{1}{b^2}-\frac{R}{b}\right). && \text{(A.16)}\end{aligned}$$

B. Solution of the equation for non-perturbative corrections in the $c = 1$ model

In this appendix we present the solution of equation (3.13). First, differentiate equation (3.13) once w.r.t. y. The result is

$$\left[2\sqrt{\alpha}\sinh\left(\frac{2-R}{2R}X\right) \mp \cos(\partial_y g)\right]\partial_y^2 g = \frac{\sqrt{\alpha}}{2}e^{\frac{1}{2}X}\partial_y X(y\partial_y - 1)g = \mp\frac{1}{2}\partial_y X\sin(\partial_y g)\,. \tag{B.1}$$

Define

$$h(X) = \pm\cos\left(\partial_y g(y)\right)\,. \tag{B.2}$$

In terms of h, equation (B.1) reads

$$\left[h - 2\sqrt{\alpha}\sinh\left(\frac{2-R}{2R}X\right)\right]\partial_X h = \frac{1}{2}(h^2-1)\,. \tag{B.3}$$

Changing variables to

$$z = \exp\left(\frac{2-R}{2R}X\right) \tag{B.4}$$

and considering z as a function of h, one arrives at a Riccati type equation

$$\partial_h z = a(h)z^2 + b(h)z + c(h)\,, \tag{B.5}$$

where

$$a(h) = -\frac{\sqrt{R-1}}{R}\frac{1}{h^2-1}\,, \qquad b(h) = \frac{2-R}{R}\frac{h}{h^2-1}\,, \qquad c(h) = \frac{\sqrt{R-1}}{R}\frac{1}{h^2-1}\,. \tag{B.6}$$

The general solution of (B.5) is easily written if a particular solution is known. One can check that

$$z_0(h) = -\sqrt{R-1}\left(h + \sqrt{h^2-1}\right) \tag{B.7}$$

is a solution. Then after the substitution

$$z(h) = z_0(h) + \frac{1}{w(h)}\,, \tag{B.8}$$

equation (B.5) takes the form

$$\partial_h w = -(2az_0 + b)w - a\,, \tag{B.9}$$

whose general solution is given by

$$w(h) = -\frac{1}{d(h)}\left(C + \int dh\, a(h)d(h)\right) \qquad \text{where } d(h) = \exp\left(\int dh\,(2az_0 + b)\right)\,. \tag{B.10}$$

The integrals can be calculated explicitly. First, one finds

$$d(h) = \sqrt{h^2-1}\,\exp\left[2\frac{R-1}{R}\log\left(h + \sqrt{h^2-1}\right)\right]\,. \tag{B.11}$$

The subsequent substitution into w gives

$$w(h) = -\frac{1}{d(h)}\left(C - \frac{1}{2\sqrt{R-1}}\exp\left[2\frac{R-1}{R}\log\left(h+\sqrt{h^2-1}\right)\right]\right). \tag{B.12}$$

Taking together (B.8), (B.7) and (B.12), one obtains

$$z(h) = -\sqrt{R-1}\left\{h+\sqrt{h^2-1}\,\frac{C'\exp\left[-2\frac{R-1}{R}\log\left(h+\sqrt{h^2-1}\right)\right]+1}{C'\exp\left[-2\frac{R-1}{R}\log\left(h+\sqrt{h^2-1}\right)\right]-1}\right\}, \tag{B.13}$$

where we redefined the integration constant $C' = 2\sqrt{R-1}C$.

The solution (B.13) is written in a form valid for $|h| > 1$. We are interested in $|h| < 1$ due to the definition (B.2). Therefore, one should replace $\sqrt{h^2-1} \to i\sqrt{1-h^2}$ in (B.13). The integration constant C' must be chosen such that $z(h)$ is real. This implies that C' is a phase, $C' = e^{2i\psi}$. Denoting

$$h = \pm\cos\phi\,, \qquad (\phi = \partial_y g(y))\,, \tag{B.14}$$

one finds[6]

$$z(\phi) = \mp\sqrt{R-1}\left\{\cos\phi + \sin\phi\tan\left(\frac{R-1}{R}\phi+\psi\right)\right\}. \tag{B.15}$$

Finally, we rewrite (B.15) as

$$z(\phi) = \mp\sqrt{R-1}\,\frac{\cos\left(\frac{1}{R}\phi-\psi\right)}{\cos\left(\frac{R-1}{R}\phi+\psi\right)}. \tag{B.16}$$

The initial condition (3.15) means that $z\,(\pi R) = 0$, and leads to

$$\psi = \frac{\pi}{2}. \tag{B.17}$$

This reduces (B.16) to (3.17). In terms of z and ϕ, our initial function g is written as follows

$$g(\phi) = y(\phi)\phi \pm \frac{1}{\sqrt{\alpha}}e^{-\frac{1}{2}X(\phi)}\sin\phi = z^{-\frac{R}{2-R}}\left((z^{-1}-z)\phi \pm \frac{1}{\sqrt{\alpha}}\sin\phi\right). \tag{B.18}$$

For the particular case $R = 3/2$ the solution can be written explicitly. Then (3.17) and (3.6) give

$$e^{\frac{1}{6}X} = \pm\sqrt{2}\cos\frac{\phi}{3}\,, \tag{B.19}$$

$$e^{-1/3X} = \frac{1}{2}(1+\sqrt{1+4y}) \tag{B.20}$$

and (B.18) leads to

$$g(y) = 3y\ \arccos\left[\pm\left(1+\sqrt{1+4y}\right)^{-1/2}\right] \pm \frac{1}{4}(1+4y)^{1/4}(3-\sqrt{1+4y}). \tag{B.21}$$

[6]The ambiguity to replace $\phi \to \phi + 2\pi n$ is captured by the integration constant ψ.

C. Disk one-point functions in $c = 1$ CFT

We want to calculate the one-point function $B_{\mathcal{T}} = \langle \cos[R(x_L - x_R)] \rangle$ of the winding one operator on the disk with Neumann boundary conditions for x. This is the same as $\langle \cos(x/R) \rangle$ with Dirichlet boundary conditions for x, which lives on a circle of radius $1/R$.

A simple way to calculate $B_{\mathcal{T}}$ is to study the annulus partition function with Neumann boundary conditions, as a function of the modulus τ in the open string channel,

$$Z_{\rm ann} = {\rm Tr}\, q^{L_0 - \frac{1}{24}}\,, \tag{C.1}$$

where $q = e^{2\pi i \tau}$ and $L_0 = p^2 + N$. The momentum p takes values in $\frac{1}{R}\mathbb{Z}$. A standard calculation gives

$$Z_{\rm ann} = \frac{1}{\eta(q)} \sum_{n \in \mathbb{Z}} q^{(n/R)^2} = \frac{\theta_3(0|\frac{2\tau}{R^2})}{\eta(\tau)}\,, \tag{C.2}$$

where $\eta(q) = q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m)$ and $\theta_3(0|\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2}$.

In the closed string channel the same annulus partition function can be represented as a sum over winding modes exchanged between two D-branes. Performing the modular transformation $\tilde{\tau} = -1/\tau$, and using the modular properties of the elliptic functions,

$$\eta(\tau) = (-i\tau)^{-1/2} \eta(-1/\tau)\,, \theta_3(0|\tau) = (-i\tau)^{-1/2} \theta_3(0| - 1/\tau)\,, \tag{C.3}$$

we find the following expansion in the closed string channel

$$Z_{\rm ann} = \frac{R}{\sqrt{2}} \frac{1}{\eta(\tilde{q})} \sum_{n \in \mathbb{Z}} \tilde{q}^{\frac{1}{2}(nR)^2}\,, \tag{C.4}$$

where $\tilde{q} = \exp(-2\pi i/\tau)$. The disk partition sum (3.35) is the square root of the $n = 0$ term in (C.4). The one point function (3.43) is the square root of the $n = 1$ term. Both are equal to

$$Z_{\rm Neumann} = B_{\mathcal{T}} = 2^{-\frac{1}{4}} \sqrt{R}\,. \tag{C.5}$$

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